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MOTION OF A HEAVY RIGID BODY ON A HORIZONTAL PLANE WITH VISCOUS FRICTION*

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The motion of an arbitrary, heavy rigid body on a horizontal plane with viscous friction is considered. It is shown that the limit set of trajectories of motion is represented by the set of motions of this body on a perfectly smooth surface without slippage. The set represents the intersection of the manifolds of steady motions of the body on perfectly smooth and perfectly rough surfaces and, depending on the dynamic and geometrical characteristics of the body, it may include the states of equilibrium, steady rotations about the vertical, uniform rolling motions along a fixed straight line, and regular processions. Examples of the motion of specific bodies are discussed.

1. Let a rigid body move along a fixed horizontal plane, touching it at a single point P of its surface. The motion takes place in a uniform gravitational field. The supporting plane is defined in the fixed $O\xi\eta\zeta$ coordinate system by the equation $\zeta = 0$, and the $O\zeta$ axis points vertically upwards. We shall introduce a right $Gxyz$, coordinate system rigidly fixed to the body, direct its axes along the principal central moments of inertia of the body, and place its origin at the centre of gravity of the body. We shall define the position of the body by the coordinates ξ, η, ζ of its centre of gravity in the fixed coordinate system, and the Euler angles ψ, θ, φ, defining the orientation of the body in absolute space. The coordinate ζ will be a known function of the angles θ and φ, i.e. $\zeta = f(\theta, \varphi) > 0$. We shall assume that the function f is a fairly smooth function of its arguments and such that the body can touch the supporting plane only at a single point of its surface. We will denote the projection of the centre of gravity G onto the supporting plane by Q. Henceforth, A, B, C will denote the moments of inertia of the body about the axes Gx, Gy, Gz, m is the mass of the body and g is the acceleration due to gravity.

We have the following expression for $\zeta^\cdot$:

$$\zeta^\cdot = \rho_\theta\theta^\cdot + \rho_\varphi\varphi^\cdot,\quad \rho_\theta = \partial f/\partial\theta,\quad \rho_\varphi = \partial f/\partial\varphi \tag{1.1}$$

The critical points of the function $f(\theta, \varphi)$ correspond to the positions of equilibrium of the body in the plane $(P = Q, \rho_\theta = \rho_\varphi = 0)$. Any body has at least two different positions of equilibrium. This follows from the fact that a function on a sphere has at least two critical points.

Let us assume that the body is acted upon at the point P from the direction of the plane by the viscous force $\mathbf{F} = -mk\mathbf{V}_P$, where $\mathbf{V}_P$ is the velocity of the point P of the body in the fixed coordinate system, and $k > 0$ is the coefficient of friction. Then the following expression can be obtained for the total energy $E > 0$ of the body:

$$dE/dt = -mkV_P^2 \tag{1.2}$$

From (1.2) we see that E does not increase and V_P tends to zero with time, i.e. the body has a tendency to avoid slipping /1/. Therefore we have

$$\lim_{t\to\infty} E(t) = E^* \geqslant \nu > 0,\quad \nu = mg\min_{\theta,\varphi} f(\theta, \varphi) \tag{1.3}$$

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If the function $f(\theta, \varphi)$ is analytic, the right-hand sides of the differential equations describing the motion of the given non-conservative system will also depend analytically on the phase coordinates. This implies that the solution will also depend analytically on the initial values and the time. Consequently, the slippage will never cease completely (although $V_P(t)$ may bome zero), otherwise $V_P(t)$ would not be an analytic function of t.

The dissipative Rayleigh function has the form $\Phi = {}^1/_2 mkV_P{}^2$; therefore the equations of motion of the body (not given in explicit form as they will not be needed) have the following property. If we put $V_P = 0$, in them, the terms depending on k will vanish. We also note that the coordinates ξ and η do not appear explicitly in the equations of motion.

Since $E > 0$, and $E^{\cdot} \leqslant 0$, the limit set Ω of the solutions of the equations of motion of a body along a plane, with viscous friction /2/, will represent the maximum invariant set contained within the bounded region $E \leqslant h$ of the phase space at whose points $E^{\cdot} = 0$. But $E^{\cdot} = 0$ if and only if $V_P = 0$. Thus the maximum invariant set of the region $E \leqslant h$, at whose points $E^{\cdot} = 0$, will be the set Ω of motions of the body in question on a perfectly smooth plane without slippage. The set Ω is asymptotically stable under any perturbations likely to arise during the motion of a body along a plane, with viscous friction, and invariant with respect to the phase flux defined by the equations of motion of the body along a plane of arbitrary roughness.

To determine the set of limit motions of the body on a plane with viscous friction, we must separte, out of all motions of this body on a perfectly smooth surface, those motions for which $V_{P\xi} = V_{P\eta} = 0$.

Let us give the expressions for the projections $V_{P\xi}, V_{P\eta}$ of the velocity of the point P of the body on the $O\xi$ and $O\eta$ axes

$$V_{P\xi} = \xi^{\cdot} + F_1 \cos\psi - F_2 \sin\psi, \quad V_{P\eta} = \eta^{\cdot} + F_1 \sin\psi + F_2 \cos\psi; \tag{1.4}$$

$$F_1 = f\varphi^{\cdot} \sin\theta + \rho_\theta (\psi^{\cdot} + \varphi^{\cdot} \cos\theta), \quad F_2 = f\theta^{\cdot} - \rho_\varphi (\psi^{\cdot} + \varphi^{\cdot} \cos\theta)/\sin\theta$$

Using (1.4) we can write the conditions of no slippage in the form

$$F_1 = -\beta \sin(\alpha + \psi), \quad F_2 = -\beta \cos(\alpha + \psi) \tag{1.5}$$

$$\beta = \sqrt{\xi^{\cdot 2} + \eta^{\cdot 2}}, \quad \sin\alpha = \xi^{\cdot}/\beta, \quad \cos\alpha = \eta^{\cdot}/\beta$$

Using (1.1) and (1.5) we obtain the following expression for $f^{\cdot}$:

$$ff^{\cdot} = -\beta\,[\rho_\varphi \sin(\alpha + \psi)/\sin\theta + \rho_\theta \cos(\alpha + \psi)] \tag{1.6}$$

and from (1.5) it follows that

$$F_1{}^2 + F_2{}^2 = \beta^2 \tag{1.7}$$

Let us consider the motion of a body on a perfectly smooth surface. The coordinates ξ, η, ψ are cyclical, and the problem reduces to the study of a Hamiltonian system with two degrees of freedom. The quantities $\xi^{\cdot}, \eta^{\cdot}$ are constant. The coordinate ψ is cyclical, and therefore relations (1.5) and (1.6) can only hold for the motions in which either $\psi^{\cdot} = 0$, or $\beta = 0$. The remaining limit motions reduce to the above motion, provided that we redefine the axes of the attached coordinate system so that they again form a right triad, or when there is dynamic symmetry, we introduce the attached $Gxyz$ coordinate system using another feasible method. Further search for the limit motions using the conditions (1.5)–(1.7), equations of motion and the first integrals, is not complicated. Only the following types of limit motions are possible.

a) The manifold Ω_1 of the positions of equilibrium of the body.

b) If one of the principal central axes of inertia of the body is orthogonal to its surface, then a manifold Ω_2 exists of the permanent rotations of the body about this axis, directed vertically. An exhaustive study of the stability of such motions is given in /3/.

c) if a cross-section of the body surface perpendicular to one of the principal central axes of inertia (e.g. Gz) contains a circumference, and the radius vector of a point lying on this circumference is orthogonal to the body surface relative to the centre of gravity G, then a manifold Ω_3 of rotations of the body exists, in which the body rolls, with this cross-section running along a fixed straight line with constant velocity. The centre of gravity lies above the point of contact. The conditions for these rotations to exist can be written analytically as follows:

$$\vartheta = \theta_0 \neq 0, \pi, \ \psi = \psi_0, \ \varphi^{\cdot} = \omega, \quad \rho_\varphi(\theta_0, \varphi) = \rho_\theta(\theta_0, \varphi) = 0$$

d) If one of the principal central axes of inertia of the body (e.g. Gz) represents the axis of dynamic symmetry, the cross-section of the body surface perpendicular to this axis contains a circumference and the radius vector of a point on this circumference is not orthogonal to the body surface relative to the centre of gravity G, then a manifold Ω_4 of regular precessions exists. The conditions for regular precessions to exist have the following analytic form /3/:

$$A = B,\ \theta = \theta_0,\ \rho_\varphi(\theta_0, \varphi) = 0,\ [Af_0 \cos\theta_0 + C(\rho_\theta \sin\theta_0 - f_0 \cos\theta_0)]\,\rho_\theta > 0$$

The stability of such motions was investigated in /3/ under the assumption that $\rho_\varphi \equiv 0$.

We note that in all motions mentioned above the height of the centre of gravity above the supporting plane is constant, i.e. $f = \mathrm{const}$.

The set $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ of limit motions of the body represents a set of all motions of the body on a plane with viscous friction, without energy dissipation. The classification given here implies that /3/ Ω lies on the intersection of the manifolds of steady motions of the body on a perfectly smooth and a perfectly rough surface. Hence, all possible steady motions are contained within Ω. The motions have been studied in some detail in /3/.

In the general case the limit motions of the body can only be represented by the positions of equilibrium, and the body has at least two positions. The position of equilibrium in which the function of height $f(\theta, \varphi)$ has a strict local minimum, is clearly Lyapunov stable. If on the other hand the function $f(\theta, \varphi)$ has no local minimum in the position of equilibrium, then the equilibrium is unstable /3/.

The structure of the set Ω, can be determined for every concrete body, taking into account its geometrical and dynamic characteristics. This means representing Ω as a union of the pairwise non-intersecting manifolds $\Omega_1, \ldots, \Omega_n$. Every manifold Ω_i will be characterized by a certain number of parameters. However, the fact that a given trajectory tends to the manifold Ω_i, does not, generally speaking, imply that it tends to a particular trajectory belonging to Ω_i. All the same, from (1.3) it follows that the total energy of the body has a limit value. Therefore, if Ω_i represents a one-parameter manifold of motions, then the fact that a trajectory tends to Ω_i, will imply that it tends to a particular trajectory of Ω_i. For example, in the case of a homogeneous triaxial ellipsoid the limit value can only be represented by a permanent rotation with specified angular velocity about one of its axes directed vertically (since only then is $f = \mathrm{const}$). The tendency of the ellipsoid to rotate about the largest axis directed vertically has been studied in /4/.

On the other hand, if the manifold Ω_i is two-parameter, condition (1.3) will enable us to determine which of the parameters has a limit value and for which "hunting" is possible. For example, for a solid of revolution for which ρ_θ vanishes in the interval $(0, \pi)$ at least once, a two-parameter manifold Ω_3 of rolling motions of the body along a straight line exists. The suitable parameters would be the angle ψ_0 and the angular velocity of rotation $\varphi^\cdot = \omega$. From (1.3) it follows that the magnitude of the angular velocity of the body tends to ω as $t \to \infty$, and though hunting is possible in ψ, its rate tends to zero.

The determination of the set Ω_i, corresponding to the given set of initial data represents a theoretically complex task. The problem can be solved e.g. on a digital computer (in the present case computing in a fairly long time interval makes it possible to determine near which invariant set Ω_i the trajectory has emerged), or using asymptotic methods.

Let the friction be small, i.e. $0 < k\sqrt{\nu/(mg^2)} \ll 1$, and let the body be dynamically and geometrically symmetric (a solid of revolution). Then $A = B$, $\rho_\varphi \equiv 0$. The equations of motion can be conveniently studied in terms of the variables $\xi^\cdot, \eta^\cdot, E, u, v, \theta, \theta^\cdot, \psi$ [5]

$$\begin{aligned}
&\xi^{\cdot\cdot} = -kV_{P\xi},\quad \eta^{\cdot\cdot} = -kV_{P\eta},\quad E^\cdot = -mkV_P^2 \\
&u^\cdot = -mk(f\sin\theta + \rho_\theta\cos\theta)(\xi^\cdot\cos\psi + \eta^\cdot\sin\psi + F_1) \\
&v^\cdot = -mk\rho_\theta(\xi^\cdot\cos\psi + \eta^\cdot\sin\psi + F_1) \\
&\theta^{\cdot\cdot} = \frac{(u - v\cos\theta)(u\cos\theta - v)}{A(A + m\rho_\theta^2)\sin^3\theta} - m\rho_\theta\frac{g + \rho_{\theta\theta}\theta^{\cdot 2}}{A + m\rho_\theta^2} - \\
&\qquad mkf\frac{-\xi^\cdot\sin\psi + \eta^\cdot\cos\psi + F_2}{A + m\rho_\theta^2} \\
&\psi^\cdot = \frac{v - u\cos\theta}{A\sin^2\theta}
\end{aligned} \tag{1.8}$$

Here v, u are projections of the kinetic momentum vector of the body about its centre of gravity, on the vertical and on the axis of symmetry of the body, and $\rho_{\theta\theta} = d^2f(\theta)/d\theta^2$. The unperturbed motion $(k = 0)$ is a motion of the solid of revolution on a smooth plane, and has been studied in some detail in /1, 6/. If we eliminate the positions of equilibrium and the motions along the separatrix, the function $\theta(t)$ will be periodic in the unperturbed motion and $\psi(t)$ will admit the representation

$$\psi(t) = \lambda_\psi t + \psi_1(t) \tag{1.9}$$

where the constant λ_ψ depends on E, u, v, and the functions $\theta(t)$ and $\psi_1(t)$ are periodic, with the same period τ, with the time-averaged value of the function $\psi_1(t)$ equal to zero.

We shall study the perturbed motion $(k \neq 0)$ using the method of averaging /7/.

In the perturbed motion the variables $E, u, v, \xi^{\cdot}, \eta^{\cdot}$ are slow, while θ and ψ are fast. In the non-resonant case, i.e. when $\lambda_\psi \neq 2\pi n/\tau$ $(n = 0, 1 \ldots)$, time avaraging can be replaced by independent averaging over ψ and over θ, where θ is a function of time, and the slow variables E, u, v. From (1.4) we see that $\langle V_{P\xi}\rangle = \xi^{\cdot}, \langle V_{P\eta}\rangle = \eta^{\cdot}$ (angle brackets denote averaging). After averaging, system (1.8) takes the form

$$\xi^{\cdot\cdot} = -k\xi^{\cdot},\quad \eta^{\cdot\cdot} = -k\eta^{\cdot},\quad E^{\cdot} = -mk(\xi^{\cdot 2} + \eta^{\cdot 2} + \langle F_1^2 + F_2^2\rangle) \tag{1.10}$$
$$v^{\cdot} = -mk\,\langle \rho_\theta F_1\rangle,\quad u^{\cdot} = -mk\,\langle (f \sin\theta + \rho_\theta \cos\theta)\, F_1\rangle$$

The first two equations of (1.10) are easily integrated, and the last three for a closed system. We find that the velocity of the centre of gravity of the body tends to zero with time, i.e. for sufficiently small k the limit set of the trajectories of motion in the non-resonant case represents the motions from Ω with zero (or almost zero) velocity of the centre of gravity. Consequently, in order for the final motion to be a motion from Ω_3, it is necessary that λ_φ and $2\pi/\tau$ be connected by the resonance relations $\lambda_\varphi = 2\pi n/\tau$ $(n = 0, 1 \ldots)$.

The limit motions of the solid of revolution will be the motions described in a) - c). This can be confirmed by equating the expressions for $\theta^{\cdot}$ obtained from (1.7), with those obtained from the expression for the total energy (given e.g. in /1/). This yields a relation connecting the first integrals E, u, v with the angle θ. It is evident that in general this relation implies that $\theta = \text{const}$. Bodies for which $\rho_\theta(\theta) = 0$ is some range of variation of the angle θ are the only exception. For such bodies rolling motions will exist during which the body will rotate uniformly about the Gx or Gy axes (see the example below).

2. We shall consider the motion, with viscous friction, of a heavy sphere on a plane. We assume that the centre of gravity of the sphere coincides with its geometrical centre and that $A > B > C$. Let G_1, G_2, G_3 be the projections of the kinetic moment G of the sphere about its centre, on the ξ, η, ζ axes respectively. The kinetic moment vector of the sphere relative to the point of contact is retained, and therefore

$$G_1 - mR\eta^{\cdot} = K_1,\quad G_2 + mR\xi^{\cdot} = K_2,\quad G_3 = K_3 \tag{2.1}$$

Here R is the radius of the sphere and $K_i = \text{const}$ $(i = 1, 2, 3)$. We shall also use the variables $\varkappa = 2T/G^2$, $1/A \leqslant \varkappa \leqslant 1/C$ (T is the kinetic energy of motion of the sphere relative to the centre of gravity).

The motion of such a sphere on a perfectly smooth plane is composed of a uniform rectilinear motion of the centre of gravity of the sphere, and of the Euler-Poinsot motion about the centre of gravity. Let us separate the motions in which $V_{P\xi} = V_{P\eta} = 0$, since $V_{P\xi} = \xi^{\cdot} - \omega_\eta R$, $V_{P\eta} = \eta^{\cdot} + \omega_\xi R$ (ω_ξ, ω_η are the projections of the vector ω on the $O\xi$ and $O\eta$) axes. Then we find at once that for such motions ω_ξ and ω_η must be constant. The geometrical Poinsot interpretation shows clearly that the motions sought can only be the rotations about one of the principal axes of inertia. This axis will have an arbitrary orientation in absolute space and the velocity components of the point P will compensate each other by virtue of the rotation and translation of the centre of gravity. Consequently, the motion of the sphere about one of the principal axes of inertia will be its limit motion. The sphere will tend asymptotically to this motion, without attaining it.

Let us direct Gz along this axes, with I denoting the moment of inertia about Gz $(I = A, B, C)$. We denote by $\theta_0, \psi_0, \xi_0^{\cdot}, \eta_0^{\cdot}, \varphi_0^{\cdot} = \omega$ the parameters of this motion. The parameters are connected with the first integral K_i in the following manner:

$$I\omega \sin\theta_0 \sin\psi_0 - mR\eta_0^{\cdot} = K_1,\quad -I\omega \sin\theta_0 \cos\psi_0 + mR\xi_0^{\cdot} = K_2,\quad I\omega \cos\theta_0 = K_3 \tag{2.2}$$

The no-slippage conditions are written in the form

$$\xi_0^{\cdot} + \omega R \sin\theta_0 \cos\psi_0 = 0,\quad \eta_0^{\cdot} + \omega R \sin\theta_0 \sin\psi_0 = 0 \tag{2.3}$$

From (2.2), (2.3) it follows that the parameters of the final motion $\theta_0, \psi_0, \omega, \xi_0^{\cdot}, \eta_0^{\cdot}$ can be expressed uniquely in terms of the first integrals K_i (the simple case $K_1 = K_2 = K_3 = 0$ is the exception). It only remains unresolved, about which axis of the body the rotation will take place (i.e. to which of the three possible values $1/A, 1/B, 1/C$, $\varkappa$ will tend).

When k are small, the method of avaraging can be used to show /8/ that for any initial conditions from the region $1/A < \varkappa < 1/B$ and for most initial conditions from the region $1/B < \varkappa < 1/C$, the sphere will tend to a rotation about the axis of the largest moment of inertia, i.e. $I = A, \varkappa \to 1/A$ as $t \to \infty$.

We also obtain the following corollary. A rolling motion of the sphere during which it rotates about the axis of greatest moment of inertia on a plane with low viscous friction, is stable relative to the variables $\theta, \theta^{\cdot}, \psi, \psi^{\cdot}, \varphi^{\cdot}, \xi^{\cdot}, \eta^{\cdot}, \varkappa$, and asymptotically stable with respect to the variable $\varkappa$. The rotations of the sphere about the axis of the smallest and mean moment of inertia are unstable.

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THE STABILITY OF NEUTRAL SYSTEMS IN THE CASE OF A MULTTPLE FOURTH-ORDER RESONANCE*

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The stability of the steady motions of multiparametric systems is investigated for the critical case of N pairs of purely imaginary roots when several internal fourth-order resonances interact with each other. The earlier investigations (see the survey /1/) covered only the interaction of odd-order resonances. In general, the problem of stability when there are even-order resonances is more complicated; even in the case of the simplest, single fourth-order resonance there is no algebraic criterion of stability /2/.

1. Consider a system of $2N$-th order for the critical case of N pairs of different, purely imaginary roots $\pm\lambda_j\ (\lambda_j^2 < 0; j = 1, \ldots, N)$, which can be written in the form /1/

$$u^{\cdot} = \lambda u + \sum_{l=2}^{\infty} U^{(l)}(u, v), \quad v^{\cdot} = -\lambda v + \sum_{l=2}^{\infty} V^{(l)}(u, v) \tag{1.1}$$
$$\lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$$

where $u = (u_1, \ldots, u_N)$, $v = (v_1, \ldots, v_N)$ are complex conjugate variables and $U^{(l)}$, $V^{(l)}$ are complex conjugate vector forms of the l-th order.

Let the first $n \leqslant N$ eigenvalues of system (1.1) be connected by $\varkappa$ fourth-order resonance relations

$$\langle P_\nu, \Lambda\rangle = 0, \nu = 1, \ldots, \varkappa \tag{1.2}$$

Here $\Lambda = (\lambda_1, \ldots, \lambda_n)$ is the eigenvalue vector and $P_\nu = (p_{\nu1}, \ldots, p_{\nu n})$is an integer-valued vector with relatively prime components, some of which may be equal to zero, and $|P_\nu| \equiv |p_{\nu1}| + \ldots + |p_{\nu n}| = 4$. We shall also assume that (1.2) does not give rise to other resonances of the same order and that there are not resonances of the order less than the fourth.

Following the generally accepted method of investigating the stability in the resonant, as well as in the non-resonant case, we will use a series of known variable substitutions /1/ to reduce system (1.1) to its normal form with an accuracy up to and including third-order terms. In the polar coordinates r_j, φ_j the system will have the form

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